IV. The Pricing Kernel and Option Pricing

Xiaoquan Liu

Department of Statistics and Finance
University of Science and Technology of China
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What is a pricing kernel?

- Liu, Shackleton, Taylor, and Xu (2009)
- Kuo, Liu, and Coakley (2010)
Option pricing

- in general we have

\[
c(K) = e^{-rT} E^Q[(S_T - K)^+]
\]

\[
= e^{-rT} \int_0^\infty (x - K)^+ f_Q(x) dx
\]

\[
= e^{-rT} \int_0^\infty (x - K)^+ \frac{f_Q(x)}{f_P(x)} f_P(x) dx
\]

\[
= \int_0^\infty (x - K)^+ m(x) f_P(x) dx
\]

\[
= E^P [m(S_T)(S_T - K)^+]
\]

- the pricing kernel, also called the stochastic discount factor, is the random variable \(m(S_T)\)

\[
m(x) = e^{-rT} \frac{f_Q(x)}{f_P(x)}
\]
In a general equilibrium, assume a representative agent faces the problem of maximizing expected utility over consumption, given budget constraint and market clearing conditions,

$$\max E_t \left[ \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \right]$$

where $c_{t+j}$ is consumption at time $t + j$ and $\beta$ is a time discount factor.
Representative agent’s problem

If we simplify the problem to a single period from \( t \) to \( t + 1 \), the problem is

\[
\max_{\theta} u(c_t) + \beta E_t[u(c_{t+1})]
\]

subject to

\[
c_t = e_t - \theta p_t \\
c_{t+1} = e_{t+1} + \theta x_{t+1}
\]

where

- \( \theta \): the holding of the asset by the agent
- \( e \): the original consumption level if nothing is traded
- \( p \): price of the asset
- \( x \): payoff of the asset
Solving the problem

Forming a Lagrangian

\[ L = u(c_t) + \beta E_t[u(c_{t+1})] + \lambda_t(e_t - c_t - \theta p_t) + \lambda_{t+1}(e_{t+1} - c_{t+1} + \theta x_{t+1}) \]

taking the first order conditions,

\[ \frac{\partial L}{\partial c_t} = 0 \Rightarrow u'(c_t) - \lambda_t = 0 \]

\[ \frac{\partial L}{\partial c_{t+1}} = 0 \Rightarrow \beta E_t[u'(c_{t+1})] - \lambda_{t+1} = 0 \]

\[ \frac{\partial L}{\partial \theta} = 0 \Rightarrow -\lambda_t p_t + \lambda_{t+1} x_{t+1} = 0 \]
The pricing kernel

- simplify to get

\[ p_t u'(c_t) = E_t[\beta u'(c_{t+1})x_{t+1}] \]

\[ p_t = E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1} \right] \]

- so the pricing kernel can also be written as

\[ m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} \]

- in discrete we have the relationship

\[ \hat{\pi}_s = \pi_s (1 + r_F) \frac{\delta u'(\tilde{c}_1 s)}{u'(c_0)} \]
The pricing kernel

- the basic pricing formula for all assets is

\[ p_t = E_t[m_{t+1}x_{t+1}] \]

or simply

\[ E[mR] = 1 \]

- the pricing kernel must be positive
- it must be downward sloping
Liu, Shackleton, Taylor, and Xu (2009)

- The pricing kernel is estimated as the ratio of RND and physical distributions.
- RND is assumed to follow:
  1. A mixture of 2 lognormal distributions.
  2. The generalized beta distribution of the 2nd kind (GB2).
  3. The cubic spline.
- Physical distribution is estimated using the GJR-GARCH model.
- The geometric mean of the ratios over time are relatively smooth; it’s generally downward sloping but there is a small hump in the middle.
Kuo, Liu, and Coakley (2010)

- specify functional form for the pricing kernel for LIBOR futures options
- follows the intertemporal CAPM of Merton (1973) and makes use of state variables, which are economic variables whose innovations can forecast future returns
- sample period is from 2000.01 to 2008.2, monthly frequency
Kuo, Liu, and Coakley (2010): Methodology

Test the following two functional forms with GMM

- A power function of returns on wealth and an exponential function of the sv

\[ m = c_0 (1 + R_W)^{-\gamma} e^{c_1 \Delta r + c_2 \Delta \eta + c_3 \Delta \sigma} \]

with moment condition

\[ E[c_0 (1 + R_W)^{-\gamma} e^{c_1 \Delta r + c_2 \Delta \eta + c_3 \Delta \sigma} (1 + \tilde{R}_{i,t+1})|I_t] = 1 \]
A sum of orthogonal Chebyshev polynomials in wealth and an exponential function of the sv (Chapman (1997) and Rosenberg and Engle (2002))

\[ m = \varphi(R_W)e^{c_1 \Delta r + c_2 \Delta \eta + c_3 \Delta \sigma} \]

\[ \varphi(R_W) = \theta_0 C_0 (1 + R_W) + \sum_{k=1}^{n} \theta_k C_k (1 + R_W) \]
To Recap

- The pricing kernel contains information essential for asset pricing \( E(mR) = 1 \).
- It can be derived as the ratio between RND and physical distribution, hence an intimate relation between the pricing kernel, RND, and physical distribution.
- Alternatively, we can assume specific functional forms that contain a measure for investor risk aversion.
- Using econometric methods, we can estimate a pricing kernel that conforms to economic theory by being everywhere positive and monotonically downward sloping.
Empirical Pricing Kernels for FTSE 100 Index Options from 1993.07 to 2003.12

![Graph showing empirical pricing kernels for FTSE 100 index options from 1993.07 to 2003.12. The graph compares two models: MLN/Historical and GB2/Historical. The x-axis represents X/F, and the y-axis represents the pricing kernel values. The graph illustrates the trend and comparison between the two models over the specified period.]
FIGURE 1. Market-related component of the pricing kernels estimated by the GMM

The pricing kernels are either with real interest rate (1 sv), with real interest rate and maximum Sharpe ratio (2 sv), or with real interest rate, maximum Sharpe ratio, and volatility (3 sv), where sv stands for state variables.
FIGURE 2. Market-related component of the pricing kernels estimated by minimizing the second HJ distance

The pricing kernels are either with real interest rate (1 sv), with real interest rate and maximum Sharpe ratio (2 sv), or with real interest rate, maximum Sharpe ratio, and volatility (3 sv), where sv stands for state variables.