III. Inferring Risk Aversion from Option Prices

Xiaoquan Liu

Department of Statistics and Finance
University of Science and Technology of China
Spring 2011
something about risk aversion coefficient
Bliss and Panigirtzoglou (2004)
Liu, Shackleton, Taylor and Xu (2007)
Absolute vs relative risk aversion coefficient

- let $u(w)$ denote von Neumann-Morgenstern utility function over wealth $w$

- the quantity $A(w) = -\frac{u''(w)}{u'(w)}$ is called the absolute risk aversion coefficient

- the quantity $R(w) = A(w)w = -\frac{wu''(w)}{u'(w)}$ is the relative risk aversion coefficient

- the most basic properties are
  1. CARA utility functions (constant absolute risk aversion) where $A(w)$ is constant
  2. CRRA utility functions (constant relative risk aversion) where $R(w)$ is constant
CARA utility

- CARA utility functions take the general form of the negative exponential
  \[ u(w) = -e^{-\rho w} \]

\[ A(w) = -\frac{u''}{u'} = -\frac{-\rho^2 e^{-\rho w}}{\rho e^{-\rho w}} = \rho \]

- individuals with CARA utility should make the same decision about risk irrespective of wealth levels, i.e. they become no less inclined or more inclined to take a given lottery as they become wealthier
CRRA utility

- utility functions that have the CRRA feature include power utility functions
- power utility is $u(w) = \frac{1}{\gamma} w^{\gamma}$ with $u'(w) = w^{\gamma-1}$ and $u''(w) = (\gamma - 1)w^{\gamma-2}$
- the ARA and RRA coefficients are
  \[ A(w) = -\frac{u''}{u'} = \frac{(1 - \gamma)w^{\gamma-2}}{w^{\gamma-1}} = \frac{1 - \gamma}{w} \]
  \[ R(w) = wA(w) = -w\frac{u''}{u'} = 1 - \gamma \]
- individuals with CRRA utility invest the same percentage of their wealth on risky assets regardless of their level of wealth
Bliss and Panigirtzoglou (2004)

- they mainly address the following questions:
  (1) are RND good forecasts of physical distributions?
  (2) what risk aversion coefficient $\gamma$ best adjusts RND to derive physical distributions?
- they use cubic spline method to derive RND, essentially a numerical method to smooth volatility smile in the volatility-delta space
- the method minimizes

$$\min_{\theta} \sum_{i=1}^{N} w_i (IV_i - IV(\Delta_i, \theta))^2 + \lambda \int_{-\infty}^{\infty} g''(x; \theta)^2 dx$$

- advantages include flexibility and control over smoothness vs accuracy ($\lambda$ controls the tradeoff)
Bliss and Panigirtzoglou (2004)

- the RND and physical distributions can be linked via utility functions
- in discrete time we have

\[ \hat{\pi}_s = \pi_s (1 + r_F) \delta u'(\tilde{c}_1s) \frac{u'(\tilde{c}_1s)}{u'(c_0)} \]

where \( s \) denotes different states of the consumption \( c \)
- the authors assume two functions forms of utility: power and exponential (see Table II)
- the power utility has constant RRA \( \gamma \) and is a popular choice in the literature
- the exponential function is more complicated and RRA is dependent on the level of asset price \( \text{RRA} = \gamma S_T \)
- hence the physical distribution can be expressed as a function of the RND and \( \gamma \)
to test the forecast ability of the RND, we have the null hypothesis that

(1) the time series of RND $\hat{f}_t(\cdot)$ is equal to the true density function $f_t(\cdot)$

(2) there is one realization $X_t$ for each option/expiry and the $X_t$ are independent

under the above null hypothesis, the inverse probability transformations of the realizations

$$y_t = \int_{-\infty}^{X_t} \hat{f}_t(u)du$$

will be independent and identically distributed $y_t \sim \text{i.i.d.} U(0, 1)$
the authors follow a test procedure in Berkowitz (2001)

1. under the null, undertake a further transformation using the inverse of the standard normal cumulative density function $\Phi(\cdot)$:
   \[
z_t = \Phi^{-1}(y_t) = \Phi^{-1} \left( \int_{-\infty}^{X_t} \hat{f}_t(u) du \right)
   \]

2. using maximum likelihood estimate the following model
   \[
z_t - \mu = \rho(z_{t-1} - \mu) + \varepsilon_t
   \]

3. under the null, the parameters of the model should be $\mu = 0$, $\rho = 0$, and $\text{Var}(\varepsilon_t) = 1$

4. denote the log-likelihood function as $L(\mu, \sigma^2, \rho)$, the likelihood ratio stat
   \[
   LR_3 = -2[L(0, 1, 0) - L(\hat{\mu}, \hat{\sigma}^2, \hat{\rho})] \text{ follows } \chi^2(3)
   \]

5. data may be overlapping; to test for independence only, the likelihood ratio stat
   \[
   LR_1 = -2[L(\hat{\mu}, \hat{\sigma}^2, 0) - L(\hat{\mu}, \hat{\sigma}^2, \hat{\rho})] \text{ follows } \chi^2(1)
   \]
If $LR_3$ rejects the null, failure to reject $LR_1$ indicates rejection of the null is due to RND being poor forecasts of physical distribution not because of lack of independence.

If both $LR_3$ and $LR_1$ reject, we cannot conclude is the problem is with poor forecast or serial correlation.

The authors select $\gamma$ value to maximize the $p-$value of the test statistic.

Main results reported in Tables IV and V.
the independence hypothesis is always accepted except for options with 6 weeks to expiry

only for the 1-week horizon, RND are good forecasts of the true physical density ($p$-value 0.233)

for the rest of maturities, markets, or utility functions, RND prove poor forecasts; this is not surprising given the pervasive risk aversion among investors hence the wedge between RND and physical distribution

for risk-adjusted densities, the null cannot be rejected for some of them (high $p$-value), including 1-week power and exponential adjusted FTSE options, 4-week power and exponential adjusted SP options, etc

there is little pattern across maturity or market or utility function
Table V

- risk aversion coefficient is significant (at different levels) for all except 6-week FTSE options
- there is some pattern of the RRA coefficient: decrease over horizon, for FTSE options the RRA from power utility adjustment are higher than those from exponential utility adjustment
- they are all at reasonable levels (see Table VII for estimates from the literature)
- there a rule of thumb: risk premium $\gamma \sigma^2$
- assume market volatility of 15% and RRA 5.10 (3-week FTSE options power utility), risk premium is 11.48%, roughly 1% per month, which is still quite high but more reasonable than most estimates from the equity market
different objective: derive closed-form transformations from RND to real-world densities (physical distributions) and show that they are better predictors than historical densities (obtained from historical asset prices via the GARCH models)

two ways of risk transformation: via power utility function and via the cdf of the Beta distribution

RND are assumed to be either a mixture of two lognormals (MLN) or follow generalized beta distribution of the 2nd kind (GB2)

the main advantages of choosing these distributions are that
(1) they are flexible and easy to estimate
(2) more importantly, both types of RND can be transformed into risk-adjusted densities in closed-form
power transformation for MLN

- the RND follows

\[ g_{MLN}(x|\theta) = w g_{LN}(x|F_1, \sigma_1, T) + (1 - w) g_{LN}(x|F_2, \sigma_2, T) \]

with parameter vector \( \theta = (F_1, F_2, \sigma_1, \sigma_2, w) \) and \( 0 \leq w \leq 1 \) and \( wF_1 + (1 - w)F_2 = F \)

- it can be shown that the real-world density is

\[ \tilde{g}(x|\theta, \gamma) = g_{MLN}(x|\tilde{\theta}) \]

with parameter vector

\[ \tilde{\theta} = (\tilde{F}_1, \tilde{F}_2, \sigma_1, \sigma_2, \tilde{w}) \]

\[ \tilde{F}_i = F_i \exp(\gamma \sigma_i^2 T), i = 1, 2 \]

\[ \frac{1}{\tilde{w}} = 1 + \frac{1 - w}{w} \left( \frac{F_2}{F_1} \right)^\gamma \exp \left( \frac{1}{2} (\gamma^2 - \gamma)(\sigma_2^2 - \sigma_1^2) T \right) \]
Liu, Shackleton, Taylor and Xu (2007)

power transformation for GB2

- the RND follows

\[ g_{GB2}(x|a, b, p, q) = \frac{a x^{ap-1}}{b^{ap} B(p, q)[1 + (\frac{x}{b})^a]^{p+q}} \]

subject to risk-neutrality constraint

\[ F = \frac{b B(p + \frac{1}{a}, q - \frac{1}{a})}{B(p, q)} \]

- it can be shown that the real-world density is

\[ \tilde{g}(x|\theta, \gamma) = g_{GB2}(x|\tilde{\theta}) \]

with real-world parameters

\[ \tilde{\theta} = \left( a, b, p + \frac{\gamma}{a}, q - \frac{\gamma}{a} \right) \]
Liu, Shackleton, Taylor and Xu (2007)

statistical calibration

- the RND $g(x)$ and real-world densities $\tilde{g}(x)$ can be shown to follow

$$\tilde{g}(x) = \frac{G(x)^{j-1}(1 - G(x))^{k-1}}{B(j, k)} g(x)$$

where $G(x)$ are cumulative distribution function of a particular $S_T$ and the Beta function $B(j, k) = \Gamma(j)\Gamma(k)/\Gamma(j + k)$

- again we obtain closed-form transformation
using non-overlapping data, the transformation parameters ($\gamma$, $j$, and $k$) are estimated using maximum log-likelihood function

$$\log(L(S_{T,1}, S_{T,2}, \cdots, S_{T,n}|\theta^*)) = \sum_{i=1}^{n} \log(\tilde{g}_i(S_{T,i}|\theta^*))$$

historical densities are simulated using parameters inferred from FTSE 100 index itself using GJR-GARCH model

data consist of non-overlapping FTSE 100 index options with 4-week to maturity from 1993.07 to 2003.12 (126 months)

RRA coefficient $\gamma$ is found to be 1.85 (MLN) and 1.86 (GB2) lower than 4.05 of Bliss and Panigirtzoglou (2004) for the same market but different market conditions (1992.06 to 2001.03); we also obtain credible estimates for $j$ and $k$

the log-likelihood is highest for risk-adjusted densities via calibration, followed by the risk-adjusted densities via power transformation, followed by the RND; the historical densities turn out to contain the least information
RND and Risk-adjusted Densities

Index Level

RND, gamma adjusted, JK adjusted
To recap

- the special characteristics of options (multiple strike prices and maturities, and forward-looking) make them an ideal asset to work with in gauging investor risk aversion
- power utility function is a popular choice in such circumstances
- risk aversion obtained from options market tend to be more credible than those inferred from equity market