II. Recovering Risk-neutral Densities from Option Prices

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risk-neutral densities (RND) and physical distributions
methods to recover RND
Melick and Thomas (1997)
Risk-neutral densities are asset price distributions in a risk-neutral world at a future point of time estimated today; also referred to as risk-neutral PDF (probability density functions).

As investors are risk-neutral, asset prices and payoffs are discounted by the riskfree rate.

In discrete time, this can be written as follows where \( \hat{\pi} \) denotes risk-neutral probabilities:

\[
p_i = \frac{1}{1 + r_F} \sum_s \hat{\pi}_s X_{is}
\]

In continuous time, option prices can be expressed as follows where \( f(S_T) \) is the RND,

\[
c(X) = e^{-r(T-t)} \int_X^{\infty} f(S_T)(S_T - X) dS_T
\]

\[
p(X) = e^{-r(T-t)} \int_{-\infty}^X f(S_T)(X - S_T) dS_T
\]
Physical distributions

- Physical distributions are also asset price distributions and ones really exist in the market.
- They contain investor risk attitude.
- They are not easy to use because we need to discount asset returns more heavily in bad states of the world but to do so we need to know investor risk aversion and risk premium.
- Compared with physical distribution, RND assigns higher probability to bad states of the world as a way to incorporate risk aversion.
RND and physical distributions for FTSE100 index options estimated on 21 March 1997

[Graph showing RND and physical distribution curves for index levels ranging from 3000 to 5500, with probability values ranging from 0 to 0.004.]

- x-axis: Index levels
- y-axis: Probability

Legend: RND (black) and Physical Distribution (red)
Breeden and Litzenberger (1978) first show that there is a one-to-one relation between option prices and RND:

\[
\frac{\partial c(X)}{\partial X} = -e^{-rT} \int_{K}^{\infty} f(S_t) dS_T
\]

\[
\frac{\partial^2 c(X)}{\partial X^2} = e^{-rT} f(X)
\]

- as proper distributions, RND must be non-negative and integrate to 1
- as risk-neutral price distributions, the mean of RND is the futures price of the asset \(F = E[S_T]\)
- ideally, there is a continuum of strike prices from 0 to infinity
- this is clearly not the case in the market so interpolation and extrapolation are needed between strike prices and outside the range of \([X_{\min}, X_{\max}]\)
because options are inherently forward-looking, they prove to be an ideal asset for estimating investor expectation of future price movements once accurately estimated, RND are very useful as they can be used to (1) price other derivatives written on the same asset; (2) hedge derivatives written on the asset; (3) used by central banks and policy-makers in adjusting interest rates, exchange rates, etc
Methods to recover RND

- Parametric methods typically assume a certain parametric function for RND, including lognormal or a mixture of lognormal functions; they have the advantage that parameters may contain economic information.

- Non-parametric methods aim to fit the distribution using polynomials or other expansions; they have the advantage that they are normally more accurate:
  - Fitting the volatility smile with polynomials (Shimko, 1993)
  - Fitting the volatility smile with cubic spline (Bliss and Panigirtzoglou, 2002)
  - Quadratic approximation (Jackwerth and Rubinstein, 1996)
  - Edgeworth expansions and hermite polynomials (Jondeau and Rockinger, 2000)

And the list goes on...
Volatility Smile

Risk-neutral Densities (USTC)
Melick and Thomas (1997)

- assume a mixture of 3 lognormal distributions as functional form for RND, which is much more flexible than the single lognormal assumption in the BS
- the RND is expressed as follows

\[ f_{\text{MLN}}(x|\theta) = \pi_1 f(x|F_1, \sigma_1, T) + \pi_2 f(x|F_2, \sigma_2, T) + \pi_3 f(x|F_3, \sigma_3, T) \]

where \( f(x|F_i, \sigma_i, T) \) is a lognormal distribution and \( \theta \) is the parameter vector
- constraints include \( \pi_1 F_1 + \pi_2 F_2 + \pi_3 F_3 = F \), \( 0 \leq \pi_i \leq 1 \), and \( \sum_{i=1}^{3} \pi_i = 1 \)
correspondingly, the option pricing formula is a mixture of three BS formula

\[ c(X|\theta, r, T) = \pi_1 c_{BS}(F_1, T, X, r, \sigma_1) + \pi_2 c_{BS}(F_2, T, X, r, \sigma_2) + \pi_3 c_{BS}(F_3, T, X, r, \sigma_3) \]

to estimate the parameters we minimize

\[ \frac{1}{N} \sum_{i=1}^{N} (c_{\text{mkt}}(X_i) - c(X_i|\theta))^2, 1 \leq i \leq N \]

where \( N \) is the number of different strike prices in a trading day

in Melick and Thomas (1997), it is slightly different as they use American options so they specify a rather tight upper and lower bounds for option prices
data are daily settlement prices for crude oil futures options from 2 July 1990 to 30 March 1991.

this sample period covers the Persian Gulf crisis between Jan to Feb 1991 (Iraq invaded Kuwait in August 1990).

during the crisis, market participants expect 3 outcomes:
(1) a return to pre-crisis conditions, e.g. Iraq withdraw peacefully from Kuwait;
(2) a severe disruption to the oil supplies, e.g. damage to Saudi oil facilities;
(3) a continuation of unsettled conditions, e.g. a prolonged stalemate in which (1) or (2) may eventually occur.

hence market participants expect oil prices to be described by a tri-modal distribution; as news hit the market, investors revise their expectation which is reflected in the shape of RND.

see Figures 7 and 8 in Section C Selected Events.
To recap

- Risk-neutral densities are future price distributions estimated today in a risk-neutral world, hence we can use them to weigh future payoff and discount at the risk-free rate.
- Options provide an ideal asset to infer RND as they are by definition forward-looking.
- A variety of methods have been proposed in the literature to estimate RND and the list is still expanding.
- Accurately inferred RND can provide important information for investors, central banks, and investment banks hence they are monitored closely.